Differential equation

(Dated: December 25, 2019)

Differential equation method for Feynman integral evaluation is a modern method for Feynman integral evaluation. The first advantage is that for numeric computation, it provides results for a list of different kinematic points.

Another advantage for the differential equation method is that the with Johannes Henn's milestone UT deferential equation [1], it is often possible to get *analytic* results for complicated loop integrals. Now it is the first option in community for Feynman integral evaluation. Only when there does not exist UT integral, we will try other methods to evaluate Feynman integrals. I believe that this is the biggest development in QFT since 2010.

Suppose that the integral family depends on Mandelstam variables s_{ij}'s and mass parameters m_i^2. Collectively, we call them x_i's. For the simplicity, we assume all of these x_i's have the same unit $[energy]^2$.

Let I be a column vector of master integrals. The derivatives of the Feynman integrals in Mandelstam variables and mass variables are again the integrals in this integral family. Therefore, by IBP reduction, we have,

$$\frac{\partial}{\partial x_i} I = A_i(x, D) I \,. \tag{1}$$

where D is the dimension of spacetime. $A_i(x, D)$ is a square matrix whose entries are rational functions in x and D.

The differential equation has the following integrability condition,

$$\frac{\partial}{\partial x_j} A_i - \frac{\partial}{\partial x_i} A_j - [A_j, A_i] = 0 \,, \tag{2}$$

otherwise there is no solution for the differential equation. Furthermore, the dimension analysis provides the Euler relation,

$$\sum_i x_i A_i = \mathrm{diag}\{[I_1]/2, \ldots, [I_k]/2\} \tag{3}$$

where $[I_j]$ is the energy dimension of I_j. The conditions (2) and (3) are very useful for checking the correctness of differential equations.

If we have a different integral basis,

$$\tilde{I} = T I \tag{4}$$

Then in the $\tilde{I}$ basis, the differential equation reads,

$$\frac{\partial}{\partial x_i}\tilde{I} = \tilde{A}_i \tilde{I}\,. \tag{5}$$

where

$$\tilde{A}_i = T A_i T^{-1} + \partial_i T T^{-1} \tag{6}$$

Note that is not a homogeneous transformation. A_i is called the *connection*, just like the "connection" in differential geometry.

I. DERIVE THE DIFFERENTIAL EQUATION

The derivative of Feynman integrals in Mandelstam variables is not straightforward, since the original Feynman integrals do not have the explicit dependence in Mandelstam variables. We need to build differential operators in external momenta which effectively serve as the derivative in Mandelstam variable.

Take the four-point massless kinematics as an example. We have $p_1^2 = p_2^2 = p_4^2 = 0$, $p_1 \cdot p_2 = s/2$, $p_1 \cdot p_4 = t/2$ and $p_2 \cdot p_4 = -(s+t)/2$. We make an ansatz for the ∂/∂_t operator,

$$\mathcal{O}_t = (c_{11}p_1^\mu + c_{12}p_2^\mu + c_{14}p_4^\mu)\frac{\partial}{\partial p_1^\mu} + (c_{21}p_1^\mu + c_{22}p_2^\mu + c_{24}p_4^\mu)\frac{\partial}{\partial p_2^\mu} + (c_{41}p_1^\mu + c_{42}p_2^\mu + c_{44}p_4^\mu)\frac{\partial}{\partial p_4^\mu} \tag{7}$$

We require that

$$\mathcal{O}_t(p_1^2) = \mathcal{O}_t(p_2^2) = \mathcal{O}_t(p_4^2) = 0, \quad \mathcal{O}_t(p_1 \cdot p_2) = 0, \quad \mathcal{O}_t(p_1 \cdot p_4) = \frac{1}{2}, \quad \mathcal{O}_t(p_2 \cdot p_4) = -\frac{1}{2} \tag{8}$$

The solution for the c's is not unique. However, the different construction are physically equivalent since the integrals themselves are Lorentz invariant. One simple construction is that,

$$\partial/\partial t = \mathcal{O}_t = \left(\frac{1}{2t}p_1^\mu + \frac{1}{2(s+t)}p_2^\mu + \frac{s+2t}{2t(s+t)}p_4^\mu\right)\frac{\partial}{\partial p_4^\mu} \tag{9}$$

Note that there is only derivative in the fourth momentum. Similarly,

$$\partial/\partial s = \mathcal{O}_s = \left(\frac{1}{2s}p_1^\mu + \frac{2s+t}{2s(s+t)}p_2^\mu + \frac{1}{2(s+t)}p_4^\mu\right)\frac{\partial}{\partial p_2^\mu} \tag{10}$$

Then acting on propagators, we get the power increase of the propagator and also a new numerator factor. Then we explicitly see that the derivatives are still in the original integral family.

Another way is to directly consider the non-vector form of the Feynman integrals. For example, it is straightforward to get the derivatives in the Baikov representation [2].

$$\partial_x G[n_1, \ldots n_k] = \partial_x\left(U^{\frac{E-D+1}{2}}\int dz_1 \ldots dz_k \frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1}\ldots z_k^{n_k}}\right) \tag{11}$$

where x is a Mandelstam variable or an external mass square. The only term looks strange is

$$\partial_x P^{\frac{D-L-E-1}{2}} \tag{12}$$

for which the degree seems to be changed. However, we proved that in ref. [2] that,

$$\partial_x P = \left(\sum_{i=1}^{k} a_k \frac{\partial}{\partial z_i} P \right) + bP \tag{13}$$

for any Feynman diagram. This identity is a statement of ideal membership. After using this relation and IBPs, we see that

$$\begin{aligned} \partial_x \left(\int dz_1 \ldots dz_k \frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1} \ldots z_k^{n_k}} \right) &= \int dz_1 \ldots dz_k \left(\sum_{i=1}^{k} \frac{a_k}{z_1^{n_1} \ldots z_k^{n_k}} \frac{\partial}{\partial z_i} P^{\frac{D-L-E-1}{2}} \right) + b \frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1} \ldots z_k^{n_k}} \\ &= \int dz_1 \ldots dz_k P^{\frac{D-L-E-1}{2}} \left(\frac{b}{z_1^{n_1} \ldots z_k^{n_k}} - \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \frac{a_k}{z_1^{n_1} \ldots z_k^{n_k}} \right) \end{aligned} \tag{14}$$

However, although this method does not have the differential operator construction step, it is in general generating more complicated expressions. Currently, its usage is limited to some special examples.

Notice that the derivatives in internal mass parameters are simple.

We consider the massless box example with

$$D_1 = l^2, \quad D_2 = (l - p_1)^2, \quad D_3 = (l - p_1 - p_2)^2, \quad D_4 = (l + p_4)^2 . \tag{15}$$

We select the master integrals

$$I = \{G[1,1,1,1], G[1,0,1,0], G[0,1,0,1]\} \tag{16}$$

It is clear that

$$\partial_t I_1 = \frac{G[0,1,1,2]}{2(s+t)} + \frac{sG[1,0,1,2]}{2t(s+t)} + \frac{G[1,1,0,2]}{2(s+t)} - \frac{(s+2t)G[1,1,1,1]}{2t(s+t)} - \frac{sG[1,1,1,2]}{2(s+t)} \tag{17}$$

$$= \frac{-6s + Ds - 2t}{2t(s+t)} G[1,1,1,1] - \frac{2(D-3)}{st(s+t)} G[1,0,1,0] + \frac{2(D-3)}{t^2(s+t)} G[0,1,0,1] \tag{18}$$

where in the second step we used the IBP reduction.

Finally we get the differential equation matrices,

$$A_s = \begin{pmatrix} -\frac{-Dt+2s+6t}{2s(s+t)} & \frac{2(D-3)}{s^2(s+t)} & -\frac{2(D-3)}{st(s+t)} \\ 0 & \frac{D-4}{2s} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

and similarly,

$$A_t = \begin{pmatrix} \frac{Ds-6s-2t}{2t(s+t)} & -\frac{2(D-3)}{st(s+t)} & \frac{2(D-3)}{t^2(s+t)} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{D-4}{2t} \end{pmatrix} \tag{20}$$

It is easily to check the integrability condition

$$\partial_t A_s - \partial_s A_t - [A_t, A_s] = 0 \tag{21}$$

and the Euler condition,

$$sA_s + tA_t = \begin{pmatrix} \frac{D-8}{2} & 0 & 0 \\ 0 & \frac{D-4}{2} & 0 \\ 0 & 0 & \frac{D-4}{2} \end{pmatrix} \tag{22}$$

We have the following observations:

- The differential equation has poles in s, t and $s+t$. However, we know that the bubble diagrams depend on s and t respectively. The box function would have singularities if $s \to 0$ or $t \to 0$, but should not have singularity if $s+t \to 0$. In future we see that the apparent $s+t$ pole would not get into the solutions (Feynman integrals).

- There are double poles. Usually double pole in a differential equation is a bad sign, which means the solution would have intrinsic singular points. A Feynman integral should not have intrinsic singular point in kinematics points. However, after a basis change this double pole is gone.

II. DIFFERENTIAL EQUATIONS FOR UT FEYNMAN INTEGRALS

From ref. [1], for a large class of Feynman integral families, we can define the uniformly transcendental (UT) integrals and the corresponding differential equation is extremely simply simple. Often, we can solve these UT differential equations *analytically.*

A list of integrals, I_i's, in one integral family, is called UT or of uniform transcendental weights, if and only if, the integrals have the following expansion in ϵ:

$$I_i = \epsilon^{-m} \sum_{j=0}^{\infty} I_i^{(j)} \epsilon^j \tag{23}$$

where m is a fixed integer and $I_i^{(j)}$ is a *pure* function of the transcendental weight j. Here we describe the meaning of transcendental weight.

We define the transcendental weight for the following quantities as,

$$\mathcal{T}(\text{rational number}) = 0, \quad \mathcal{T}(\pi) = 1, \quad \mathcal{T}(\zeta_n) = n \tag{24}$$

$$\mathcal{T}(\text{rational function}) = 0, \quad \mathcal{T}(\log(\ldots)) = 1, \quad \mathcal{T}(\mathrm{Li}_n(\ldots)) = n \tag{25}$$

where $\mathrm{Li}_n(\ldots)$ is the order n polylogarithm function.

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad \mathrm{Li}_1(z) = -\log(1-z) \tag{26}$$

Furthermore, we define the transcendental weight of a product as,

$$\mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)\,. \tag{27}$$

$I_i^{(j)}$ is a function of the transcendental weight j, if for all terms of $I_i^{(j)}$, the transcendental weight is j, $\mathcal{T}(I_i^{(j)}) = j$. Furthermore $I_i^{(j)}$ is pure if and only if

$$\mathcal{T}(\partial_x I_i^{(j)}) = j - 1, \tag{28}$$

for any kinematic variable.

If I_i is UT and also an integral basis, then its differential equation is very simple [1],

$$\frac{\partial}{\partial x_i} I = \epsilon A_i(x) I \tag{29}$$

where $A_i(x)$ is rational and independent of ϵ. This kind of DE is called the canonical DE. The integrability condition is simplified as,

$$\partial_j A_i = \partial_i A_j, \quad [A_i, A_j] = 0 \tag{30}$$

Immediately, we see that the differential matrices are all integrated to *the same matrix* A:

$$A_i = \partial_i A\,. \tag{31}$$

Note that A may not be extended to a global single-valued matrice but locally it is well-defined. We further simplify (30) as,

$$dI = \epsilon (dA) I \tag{32}$$

here d stands for the total derivative. If we consider a path $\gamma(z)$ which maps $[0,1]$ to the parameter space, then along the path

$$\frac{d}{dz} I = \epsilon \frac{dA}{dz} I \tag{33}$$

Now consider I along the path γ, namely $I(z)$. Suppose that the boundary at the boundary $z = 0$, $I(0)$ is known to certain orders,

$$I_i(0) = \epsilon^{-k} \sum_{j=0} B_i^{(j)} \epsilon^j \tag{34}$$

where $B_i^{(j)}$ are constants. Then the canonical DE is solved immediately by the path-ordered exponential expansion, just like the perturbation series in QFT,

$$I(z) = \mathcal{P} \exp\left(\epsilon \int_0^z \omega(u) du\right) I(0) \,. \tag{35}$$

where $\omega(u) = dA/dz|_{z=u}$. That means order by order,

$$I^{(0)}(z) = B^{(0)} \tag{36}$$

$$I^{(1)}(z) = B^{(1)} + \int_0^z du \; \omega(u) B^{(0)} \tag{37}$$

$$I^{(2)}(z) = B^{(2)} + \int_0^z du \; \omega(u) B^{(1)} + \int_0^z du_1 \int_0^{u_2} du_2 \; \omega(u_1)\omega(u_2) B^{(0)} \tag{38}$$

$$\begin{aligned} I^{(3)}(z) = B^{(3)} &+ \int_0^z du \; \omega(u) B^{(2)} + \int_0^z du_1 \int_0^{u_1} du_2 \; \omega(u_1)\omega(u_2) B^{(1)} \\ &+ \int_0^z du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \; \omega(u_1)\omega(u_2)\omega(u_3) B^{(0)} \end{aligned} \tag{39}$$

$$\ldots \tag{40}$$

So each order is explicitly an iterative integral. From the definition of transcendental weights, we see that if the DE is canonical and the boundary is UT, then the integral basis is UT. However, if only the DE is canonical we conclude that *very likely* the integral basis is UT but there can be counter-examples.

Does the result depend on the choice of γ? The integrability condition implies that the iterative integral is homotopically invariant. However a non-homotopic choice will provides different integration result. That means the Feynman integrals are multi-valued functions but holomorphic on each branch.

In general the total derivative matrix dA has the *symbol letter* decomposition.

$$dA = \sum_{k=1}^{N} c_k \frac{dW_k}{W_k} \tag{41}$$

where c_k is a constant square matrix and W_k is a function of kinematic variables.

Then we see that on the curve,

$$\omega(z) = \frac{dA}{dz} = \sum_{k=1}^{N} c_k \frac{dW_k}{dz} \frac{1}{W_k} = \sum_{k=1}^{N} c_k \frac{d \log W_k}{dz} \tag{42}$$

So we are working with the iterative integrals with dlog forms.

A. Example: massless one-loop box

We compute the massless one-loop box explicitly by UT integrals.

First, to simplify the problem, we remove the dimension by,

$$x = s/t \tag{43}$$

And set the UT integrals as

$$I_1 = \epsilon^2 e^{\epsilon \Gamma_E} (-s)^\epsilon st G[1,1,1,1] \tag{44}$$

$$I_2 = \epsilon e^{\epsilon \Gamma_E} (-s)^\epsilon s G[1,0,2,0] \tag{45}$$

$$I_3 = \epsilon e^{\epsilon \Gamma_E} (-s)^\epsilon t G[1,0,2,0] \tag{46}$$

The factor s^ϵ is set to make all these UT integrals dimensionless.

Then the differential equation in x reads

$$\partial_x I = \epsilon \begin{pmatrix} -\frac{1}{x(x+1)} & \frac{2}{x+1} & -\frac{2}{x(x+1)} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{x} \end{pmatrix} I = \left(\begin{pmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{x+1} \right) I \tag{47}$$

We call the residue matrix at 0 as A_0 and the residue matrix at 1 as A_{-1}.

From direct computation in Feynman representation, we see that

$$I_2 = \frac{\pi e^{\gamma \epsilon} \epsilon^2 \csc(\pi \epsilon) \Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \tag{48}$$

and $I_3 = x^\epsilon I_2$.

We choose the boundary point at $x = 1$. At this point, these is no singularity. This is a symmetric point for I_2 and I_3. The boundary values are,

$$I|_{z=1} = \begin{pmatrix} b_0 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} \epsilon + \begin{pmatrix} b_2 \\ \frac{\pi^2}{12} \\ \frac{\pi^2}{12} \end{pmatrix} \epsilon^2 + \begin{pmatrix} b_3 \\ \frac{7\zeta(3)}{3} \\ \frac{7\zeta(3)}{3} \end{pmatrix} \epsilon^3 + \begin{pmatrix} b_4 \\ \frac{47\pi^4}{1440} \\ \frac{47\pi^4}{1440} \epsilon^4 \end{pmatrix} \tag{49}$$

All these b_i's are unknown which are to be fixed.

Using the iterative integral, first, we find,

$$I^{(1)} = B^{(1)} + \int_1^x dz (\frac{A_0}{z} + \frac{A_1}{1+z}) I^{(0)} \tag{50}$$

It seems that we will get $\log(x+1)$ after the integration. However, this term violates that condition that the box integral should not have singularity at $x = -1$. That means

$$A_{-1} I^{(0)} = 0, \quad b_0 = 4 \tag{51}$$

So after integration we have

$$I^{(1)} = \begin{pmatrix} b_1 - 2\log(x) \\ 0 \\ \log(x) \end{pmatrix} \tag{52}$$

Repeat this,

$$I^{(2)} = B^{(2)} + \int_1^x dz\left(\frac{A_0}{z} + \frac{A_1}{1+z}\right)I^{(1)} \tag{53}$$

Again we need $A_1 I^{(1)}|_{z\to -1}$ vanishes. So $b_1 = 0$. After the integration

$$I^{(2)} = \begin{pmatrix} b_2 \\ \frac{\pi^2}{12} \\ \frac{\pi^2}{12} - \frac{1}{2}\log^2(x) \end{pmatrix} \tag{54}$$

Then

$$I^{(3)} = B^{(3)} + \int_1^x dz\left(\frac{A_0}{z} + \frac{A_1}{1+z}\right)I^{(2)} \tag{55}$$

From the singularity analysis at -1, we determine that $b_2 = -4\pi^2/3$. After the integration,

$$I^{(3)} = \begin{pmatrix} b_3 + \frac{\log^3(x)}{3} - \log(x+1)\log^2(x) - 2\mathrm{Li}_2(-x)\log(x) + \frac{7}{6}\pi^2\log(x) + \pi^2\log\left(\frac{2}{x+1}\right) + 2\mathrm{Li}_3(-x) + \frac{3\zeta(3)}{2} \\ \frac{7\zeta(3)}{3} \\ \frac{1}{12}\left(2\log^3(x) - \pi^2\log(x) + 28\zeta(3)\right) \end{pmatrix} \tag{56}$$

Then in the next integration we find that $b_3 = -77\zeta(3)/6$. Thus,

$$I^{(3)} = \begin{pmatrix} \frac{\log^3(x)}{3} - \log(x+1)\log^2(x) - 2\mathrm{Li}_2(-x)\log(x) + \frac{7}{6}\pi^2\log(x) + \pi^2\log\left(\frac{2}{x+1}\right) + 2\mathrm{Li}_3(-x) - \frac{34\zeta(3)}{3} \\ \frac{7\zeta(3)}{3} \\ \frac{1}{12}\left(2\log^3(x) - \pi^2\log(x) + 28\zeta(3)\right) \end{pmatrix} \tag{57}$$

Note that this function is finite at $x = -1$. In this way, all order in the ϵ is found analytically.

---

[1] J. M. Henn, Phys. Rev. Lett. **110**, 251601 (2013), 1304.1806.

[2] J. Bosma, K. J. Larsen, and Y. Zhang, Phys. Rev. **D97**, 105014 (2018), 1712.03760.